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DIFFRACTION OF ACOUSTIC WAVES IN A PLANE SEMI-INFINITE WAVEGUIDE WITH ELASTIC WALLS

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Exact analytic expression for Green's function of the Helmholtz equation for the half-strip and boundary conditions that contain high order derivatives is obtained by the method of expansion in terms of plane waves. This problem arises in the determination of the acoustic field created by a point source in a plane semi-infinite acoustic waveguide with thin elastic walls, and also inside an infinite acoustic waveguide with a thin elastic baffle.

1. Statement of the problem. Examples. We seek the solution of the problem

$$(\Delta + k^2)\, P(x, y) = -\delta(x - x_0,\ y - y_0),\quad 0 < x < \infty,\ 0 < y < h \tag{1.1}$$

$$L_\alpha P(x, y_\alpha) = 0,\quad 0 < x < \infty \quad \alpha = 1, 2;\quad y_1 = 0,\quad y_2 = h \tag{1.2}$$

$$L_3 P(0, y) = 0,\quad 0 < y < h \tag{1.3}$$

$$L_\alpha = (-1)^{\alpha+1} m_{\alpha 1}\left(-\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial y} + m_{\alpha 2}\left(-\frac{\partial^2}{\partial x^2}\right),\quad \alpha = 1, 2$$

$$L_3 = m_{31}\left(-\frac{\partial^2}{\partial y^2}\right)\frac{\partial}{\partial x} + m_{32}\left(-\frac{\partial^2}{\partial y^2}\right)$$

where P is the acoustic pressure in the medium, Δ is the Laplace operator, k is the wave number, the time dependence is specified by the factor $e^{-i\omega t}$ which is omitted throughout, $m_{\alpha\beta}$ are polynomials of their arguments whose coefficients are independent of space coordinates x and y. In the considered region the sought solution must be continuous up to the boundary, with the exception of point (x_0, y_0) of location of the source, and must satisfy the principle of ultimate absorption.

For the simplest Dirichlet or Neumann boundary conditions the considered problem has a unique solution whose derivation by the method of images is elementary. For boundary operators of order higher than unity the solution loses its uniqueness and contains N arbitrary constants. The number of these constants can be determined by formulas [1, 2]

$$N = N^{(1)} + N^{(2)} \tag{1.4}$$

$$N^{(1)} = E\left(\frac{N_1 + N_3 - 1}{2}\right), \quad N^{(2)} = E\left(\frac{N_2 + N_3 - 1}{2}\right)$$

where N_α $(\alpha = 1, 2, 3)$ is the differential order of operator L_α and $E(x)$ denotes the whole part of number x.

The arbitrariness of solution is eliminated by supplementing the statement of the problem by N boundary-contact conditions that specify the mechanical mode at conditional points of region

$$(R_\beta{}^\alpha + S_\beta{}^\alpha)\, P(0, y_\alpha) = 0 \tag{1.5}$$

$$R_\beta{}^\alpha P(0, y_\alpha) = \lim_{x\to 0}\left[(-1)^{\alpha+1} r_{\beta 1}^\alpha\left(-i\frac{\partial}{\partial x}\right)\frac{\partial}{\partial y} + r_{\beta 2}^\alpha\left(-i\frac{\partial}{\partial x}\right)\right]P(x, y_\alpha)$$

$$S_\beta{}^\alpha P(0, y_\alpha) = \lim_{y\to y_\alpha}\left[\left(-i\frac{\partial}{\partial x}\right)s_{\beta 1}^\alpha\left(-\frac{\partial}{\partial y}\right) + s_{\beta 2}^\alpha\left(-\frac{\partial}{\partial y}\right)\right]P(0, y)$$

$$\beta = 1, 2, \ldots, N^{(\alpha)}, \qquad \alpha = 1, 2$$

where $r_{\beta\delta}{}^\alpha$ and $s_{\beta\delta}{}^\alpha$ $(\gamma = 1, 2)$ are polynomials of their arguments.

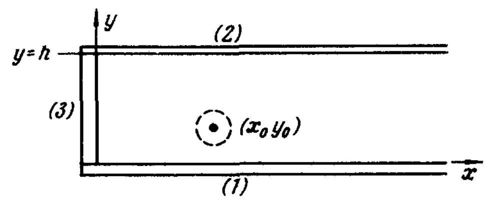

Fig. 1

Example 1. The field of a point source in a semi-infinite waveguide with walls in the form of elastic plates whose motions are purely flexural is defined by Fig. 1.

$$L_\alpha = (-1)^{\alpha+1}\left(\frac{\partial^4}{\partial x^4} - k_\alpha{}^4\right)\frac{\partial}{\partial y} + \nu_\alpha, \quad \alpha = 1, 2 \tag{1.6}$$

$$L_3 = \left(\frac{\partial^4}{\partial y^4} - k_3{}^4\right)\frac{\partial}{\partial x} + \nu_3, \quad \nu_\alpha = \frac{\rho\omega^2}{D_\alpha}$$

where k_α are wave numbers of flexural waves in the plates, ρ is the density of the acoustic medium, and D_α is the torsional rigidity of plates $(\alpha = 1, 2, 3)$.

We assume that the plates are rigidly connected to each other at points $(0, y_\alpha)$. In that case the boundary-contact conditions are

$$\lim_{x\to 0}\frac{\partial}{\partial y}P(x, y_\alpha) = \lim_{y\to y_\alpha}\frac{\partial}{\partial x}P(0, y) = 0 \tag{1.7}$$

$$\lim_{x\to 0}\frac{\partial^2}{\partial x\,\partial y}P(x, y_\alpha) + \lim_{y\to y_\alpha}\frac{\partial^2}{\partial y\,\partial x}P(0, y) = 0 \tag{1.8}$$

$$(-1)^\alpha D_\alpha \lim_{x\to 0}\frac{\partial^3}{\partial x^2\,\partial y}P(x, y_\alpha) + D_3\lim_{y\to y_\alpha}\frac{\partial^3}{\partial y^2\,\partial x}P(0, y) = 0 \quad (\alpha = 1, 2) \tag{1.9}$$

Condition (1.7) implies the absence of displacements at plate connections, (1.8) implies the invariance of the angle between plates, and (1. 9) implies the absence of external torsional moments at plate connections. The over-all number of boundary-contact conditions in in conformity with (1. 4) equal eight.

Example 2. The field of a point source in an infinite waveguide with elastic walls is covered by an elastic baffle rigidly connected to the walls. As in Example 1 we assume the motions of plates to be purely flexural. We have to find the solution of Eq. (1. 1) for $x \neq 0$ and $0 < y < h$ which satisfies boundary conditions (1. 2) for $x \neq 0$ and the matching conditions for $x = 0$ and $0 < y < h$ which define the continuity of normal displacements at the plate-acoustic medium interface and the balance of forces acting on the plate

$$\frac{\partial}{\partial x} P(-0, y) = \frac{\partial}{\partial x} P(+0, y)$$

$$\frac{1}{2}\left(\frac{\partial^4}{\partial y^4} - k_3^4\right)\left[\frac{\partial}{\partial x} P(-0, y) + \frac{\partial}{\partial x} P(+0, y)\right] = \nu_3 [P(-0, y) - P(+0, y)]$$

After separation of the acoustic field into even and odd parts with respect to the variable x, the problem reduces to two independent problems of the considered type.

2. Solution of the problem. The solution derived below satisfies all requirements of the problem with the exception of the boundary-contact conditions. It is sought in the form

$$P(x, y) = P_0(x, y) + P^*(x, y) \tag{2.1}$$

$$P_0(x, y) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \exp(i\lambda(x - x_0) - \gamma|y - y_0|) \frac{d\lambda}{\gamma} \tag{2.2}$$

where function $P_0(x, y)$ represents the field of a point source at coordinates (x_0, y_0) in an unbounded medium. The notation $\gamma = \sqrt{\lambda^2 - k^2}$, is used in (2. 2) and the choice of the radical branch is fixed by the requirement that $\operatorname{Re}\gamma > 0$ for $\operatorname{Im}\lambda = 0$ and $\operatorname{Im} k > 0$.

Because the field of source P_0 is specified by different formulas for $y < y_0$ and $y > y_0$, it is convenient to seek P^* of the form

$$P^*(x, y) = \begin{cases} P_1(x, y), & 0 < y < y_0 \\ P_2(x, y), & y_0 < y < h \end{cases} \tag{2.3}$$

$$P_\alpha(x, y) = \frac{1}{4\pi} \int_\Lambda [p_\alpha(\lambda) e^{\gamma y} + q_\alpha(\lambda) e^{-\gamma y}] e^{i\lambda x} \frac{d\lambda}{\gamma} \tag{2.4}$$

where p_α and q_α are unknown functions of the complex variable λ, Λ is the continuous integration contour passing along the real axis of the complex plane from $-\infty$ to $+\infty$, with the exception of some finite section. The behavior of the contour in that section is described below.

Functions P_α^* must in addition satisfy for $y = y_0$, $0 < x < \infty$ and $x \neq x_0$ the matching conditions

$$P_1^*(x, y_0) = P_2^*(x, y_0), \quad \frac{\partial}{\partial y} P_1^*(x, y_0) = \frac{\partial}{\partial y} P_2^*(x, y_0) \tag{2.5}$$

With the use of the boundary conditions (1. 2) and of matching conditions (2. 5) we obtain the following system of integral equations that must be satisfied by the unknown functions:

$$\int\limits_{\Lambda} [l_2 p_2 e^{\gamma h} + l_2^{\circ} q_2 e^{-\gamma h} + l_2^{\circ} e^{-i\lambda x_0 - \gamma(h-y_0)}] e^{i\lambda x} \frac{d\lambda}{\gamma} = 0 \qquad (2.6)$$

$$\int\limits_{\Lambda} [l_1^{\circ} p_1 + l_1 q_1 + l_1^{\circ} e^{-i\lambda x_0 - \gamma y_0}] e^{i\lambda x} \frac{d\lambda}{\gamma} = 0 \quad (x > 0)$$

$$\int\limits_{\Lambda} [(p_2 - p_1) e^{\gamma y_0} + (q_2 - q_1) e^{-\gamma y_0}] e^{i\lambda x} \frac{d\lambda}{\gamma} = 0$$

$$\int\limits_{\Lambda} [(p_2 - p_1) e^{\gamma y_0} - (q_2 - q_1) e^{-\gamma y_0}] e^{i\lambda x} d\lambda = 0$$

where

$$l_\alpha(\lambda) = -\gamma m_{\alpha 1}(\lambda^2) + m_{\alpha 2}(\lambda^2)$$
$$l_\alpha^{\circ}(\lambda) = \gamma m_{\alpha 1}(\lambda^2) + m_{\alpha 2}(\lambda^2), \quad \alpha = 1, 2$$

Integral equations (2.6) are satisfied if we set

$$l_2 p_2 e^{\gamma h} + l_2^{\circ} q_2 e^{-\gamma h} + l_2^{\circ} e^{-i\lambda x_0 - \gamma(h-y_0)} = \gamma\, \Phi_2^+(\lambda) \qquad (2.7)$$
$$l_1^{\circ} p_1 + l_1 q_1 + l_1^{\circ} e^{-i\lambda x_0 - \gamma y_0} = \gamma \Phi_1^+(\lambda)$$
$$(p_2 - p_1) e^{\gamma y_0} + (q_2 - q_1) e^{-\gamma y_0} = 2\gamma A_+(\lambda)$$
$$(p_2 - p_1) e^{\gamma y_0} - (q_2 - q_1) e^{-\gamma y_0} = 2B_+(\lambda)$$

where Φ_α^+, A_+ and B_+ are functions that are analytic above the contour Λ. The formula for $P = P_0 + P^*$ in terms of functions Φ_α^+, A_+ and B_+ is of the form

$$P_0(x, y) + P_2^*(x, y) = \frac{1}{4\pi} \int\limits_{\Lambda} e^{i\lambda x} \frac{d\lambda}{D(\lambda)} \{t_2(\lambda, y) \Phi_1^+(\lambda) + \qquad (2.8)$$
$$t_1(\lambda, y) \Phi_2^+(\lambda) + t_2(\lambda, y) t_1(\lambda, y_0) [e^{-i\lambda x_0} - B_+(\lambda)] +$$
$$t_2(\lambda, y) t_1'(\lambda, y_0) A_+(\lambda)\} \qquad (y_0 < y < h)$$

$$P_0(x, y) + P_1^*(x, y) = \frac{1}{4\pi} \int\limits_{\Lambda} e^{i\lambda x} \frac{d\lambda}{D(\lambda)} \{t_2(\lambda, y) \Phi_1^+(\lambda) +$$
$$t_1(\lambda, y) \Phi_2^+(\lambda) + t_1(\lambda, y) t_2(\lambda, y_0) [e^{-i\lambda x_0} - B_+(\lambda)] +$$
$$t_1(\lambda, y) t_2'(\lambda, y_0) A_+(\lambda)\} \qquad (0 < y < h)$$

where

$$\gamma D(\lambda^2) = l_1 l_2 e^{\gamma h} - l_1^{\circ} l_2^{\circ} e^{-\gamma h}$$
$$\gamma t_1(\lambda, y) = l_1 e^{\gamma y} - l_1^{\circ} e^{-\gamma y}, \quad \gamma t_2(\lambda, y) = l_2 e^{\gamma(h-y)} - l_2^{\circ} e^{-\gamma(h-y)}$$
$$t_\alpha'(\lambda, y) = \frac{\partial}{\partial y} t_\alpha(\lambda, y)$$

Using the boundary condition (1.3) we obtain the following system of integral equations:

$$\int\limits_{\Lambda} \frac{n_3(\lambda)\, d\lambda}{D(\lambda)} \{t_2(\lambda, y) \Phi_1^+(\lambda) + t_1(\lambda, y) \Phi_2^+(\lambda) + \qquad (2.9)$$
$$t_2(\lambda, y) t_1(\lambda, y_0) [e^{-i\lambda x_0} - B_+(\lambda)] + t_2(\lambda, y) t_1'(\lambda, y_0) A_+(\lambda)\} = 0$$
$$(y_0 < y < h)$$

$$\int\limits_{\Lambda} \frac{n_3(\lambda)\, d\lambda}{D(\lambda)} \{t_2(\lambda, y) \Phi_1^+(\lambda) + t_1(\lambda, y) \Phi_2^+(\lambda) + t_1(\lambda, y) t_2(\lambda, y_0) \times$$

$$[e^{-i\lambda x_0} - B_+(\lambda)] + t_1(\lambda, y)\, t_2'(\lambda, y_0)\, A_+(\lambda)\} = 0 \quad (0 < y < h)$$

Integral equations (2. 9) are satisfied, if we make the integration contour Λ symmetric about the coordinate origin and set

$$\begin{aligned} & n_3(\lambda)\, \Phi_\alpha^+(\lambda) = \lambda \Phi_\alpha(\lambda^2), \quad n_3(\lambda)\, A_+(\lambda) = \lambda A(\lambda^2) \\ & n_3(\lambda)\, B_+(\lambda) - n_3(-\lambda) e^{i\lambda x_0} = \lambda B(\lambda^2) \\ & n_3(\lambda) = i\lambda m_{31}(-\gamma^2) + m_{32}(-\gamma^2) \end{aligned} \tag{2. 10}$$

where $\Phi_\alpha(\lambda^2)$, $A(\lambda^2)$ and $B(\lambda^2)$ are even functions of the complex variable λ. In accordance with the theorem of analytic continuation they are entire functions of the complex variable λ.

For functions $P^*(x, y)$ to be continuous in region $x \geqslant 0$, $0 \leqslant y \leqslant h$ it is sufficient if in the neighborhood of an infinitely distant point of the complex plane λ the following estimates:

$$\begin{aligned} & n_3(\lambda)\, \Phi_\alpha^+(\lambda) = O(\lambda^{N_\alpha + N_3 - 1 - \varepsilon_\alpha}) \\ & n_3(\lambda)\, A_+(\lambda) = O(\lambda^{N_3 - 1 - \varepsilon_3}) \\ & n_3(\lambda)\, B_+(\lambda) - n_3(-\lambda)\, e^{i\lambda x_0} = O(\lambda^{N_3 - \varepsilon_4}) \end{aligned}$$

are satisfied.

By the Liouville theorem functions $\Phi_\alpha(\lambda^2)$, $A(\lambda^2)$ and $B(\lambda^2)$ are polynomials of the complex variable λ^2 and the power of $\Phi_\alpha(\lambda^2)$ is $N^{(\alpha)} - 1$.

Functions Φ_α^+, A_+ and B_+ must be analytic above the contour Λ. Because of this the contour Λ must be chosen so that roots of $n_3(\lambda)$ are below it [2]. Note that $L_3 P(0, y)$, where $P(x, y)$ is determined by formulas (2. 8), is a linear combination of $\delta(y - y_0)$ and of some of its derivatives. The coefficients of that combination are expressed in terms of coefficients of polynomials $A(\lambda^2)$ and $B(\lambda^2)$, which means that the part of $P(x, y)$ which depends on $A(\lambda^2)$ and $B(\lambda^2)$ defines the field of point inhomogeneity of the boundary, which arises owing to the partitioning of the region by the line $y = y_0$. Hence, to satisfy condition (1. 3) it is necessary to equate $A(\lambda^2)$ and $B(\lambda^2)$ to zero.

After transformation the expression for P assumes the form

$$P(x, y) = P_1(x, y) + P_2(x, y) + Q_1(x, y) + Q_2(x, y) \tag{2. 11}$$

$$P_1(x, y) = \frac{1}{4\pi} \int_\Lambda e^{i\lambda(x - x_0)} \frac{t(\lambda, y, y_0)}{D(\lambda)}\, d\lambda \tag{2. 12}$$

$$P_2(x, y) = -\frac{1}{4\pi} \int_\Lambda e^{i\lambda(x + x_0)} \frac{n_3(-\lambda)\, t(\lambda, y, y_0)}{n_3(\lambda)\, D(\lambda)}\, d\lambda$$

$$Q_1(x, y) = \frac{1}{4\pi} \int_\Lambda e^{i\lambda x} \frac{t_2(\lambda, y)\, \Phi_1(\lambda^2)}{n_3(\lambda)\, D(\lambda)}\, \lambda\, d\lambda$$

$$Q_2(x, y) = \frac{1}{4\pi} \int_\Lambda e^{i\lambda x} \frac{t_1(\lambda, y)\, \Phi_2(\lambda^2)}{n_3(\lambda)\, D(\lambda)}\, \lambda\, d\lambda$$

$$\gamma^2 t(\lambda, y, y_0) = l_1 l_2 e^{\gamma(h - |y - y_0|)} + l_1^\circ l_2^\circ e^{-\gamma(h - |y - y_0|)} - l_1^\circ l_2 e^{\gamma(h - y - y_0)} - l_1 l_2^\circ e^{-\gamma(h - y - y_0)}$$

where P_1 is the field of the point source in an infinite waveguide, P_2 is the field of the source image relative to the boundary $x = 0$, and Q_α is the field that radiates from angle points $(0,\ y_\alpha)$ of the region.

The integrands in these expressions do not have branching points for $\lambda = \pm k$ and are fractional functions of the complex variable λ. Poles of these functions lie at points λ_s at which $D(\lambda_s) = 0$, as well as at the roots of polynomial $n_3(\lambda)$.

Roots of $D(\lambda)$ represent wave numbers of normal waves. Owing to the evenness of $D(\lambda)$ wave numbers of normal waves are symmetric about $\lambda = 0$. For $\operatorname{Im} k > 0$ $D(\lambda)$ has no real roots, and the contour Λ lies along the real axis, with the exception of the neighborhood of the coordinate origin, where, owing to its symmetry about $\lambda = 0$, the roots of $n_3(\lambda)$ and of $D(\lambda)$ for which $0 < \arg \lambda_s < \pi$ remain, respectively, below and above it. We assume that none of the roots of $n_3(\lambda)$ coincide with any of the roots of $D(\lambda)$ and $n_3(-\lambda)$. For $\operatorname{Im} k = +0$ some roots of $D(\lambda)$ appear on the real axis, with the roots from the upper and lower half-planes situated on the positive and negative parts of that axis, respectively. Thus in the absence of absorption Λ is displaced from the real axis and bypasses positive roots of $D(\lambda)$ from below and negative roots from above. Integration contours of the convergent kind had already occurred in the analysis of acoustic wave diffraction on plates joined at right angle [2].

In the absence of absorption, a certain finite number of roots appear in the medium and at walls of the waveguide in the section $(-k,\ k)$. The number of these increases with the waveguide dimensionless width kh. A denumerable set of roots is found along the imaginary axis for considerable natural s asymptotically arranged at points $\pm i\pi s / h$. Finally, there exists a finite set of roots which, with increasing kh, convert into those roots $l_\alpha(\lambda)$ for which $\operatorname{Re}\gamma > 0$. The number of these roots is determined by the properties of differential operators L_α and is independent of the waveguide width kh. Real roots lying in segments $(-\infty,\ -k)$ and $(k,\ \infty)$ may be found among these. Such roots determine the wave numbers of "boundary layer" waves whose amplitude decreases exponentially with increasing distance from the waveguide walls. For boundary operators defined by formulas (1. 6) there are two such waves. They are associated with the flexural motions of each wall of the waveguide.

Integration in (2. 12) can be reduced to summation of a series in residues, which represents the superposition of normal waves

$$P_1(x,\ y) = \frac{i}{2}\sum_s e^{i\lambda_s|x-x_0|}\frac{t_1(\lambda_s,\ y_0)\, t_2(\lambda_s,\ y)}{D'(\lambda_s)} \tag{2. 13}$$

$$P_2(x,\ y) = -\frac{i}{2}\sum_s e^{i\lambda_s(x+x_0)}\,\frac{n_3(-\lambda_s)\, t_1(\lambda_s,\ y_0)\, t_2(\lambda_s,\ y)}{n_3(\lambda_s)\, D'(\lambda_s)}$$

$$Q_1(x,\ y) = \frac{i}{2}\sum_s e^{i\lambda_s x}\,\frac{\lambda_s t_2(\lambda_s,\ y)\,\Phi_1(\lambda_s^2)}{n_3(\lambda_s)\, D'(\lambda_s)}$$

$$Q_2(x,\ y) = \frac{i}{2}\sum_s e^{i\lambda_s x}\,\frac{\lambda_s t_1(\lambda_s,\ y)\,\Phi_2(\lambda_s^2)}{n_3(\lambda_s)\, D'(\lambda_s)}\quad \left(D'(\lambda) = \frac{d}{d\lambda}D(\lambda)\right)$$

where summation is extended to the roots of $D(\lambda)$ that lie above Λ.

Note that the equality

$$t_1(\lambda_s,\ y_0)\, t_2(\lambda_s,\ y) = t_1(\lambda_s,\ y)\, t_2(\lambda_s,\ y_0)$$

which indicates the symmetry of Green's function P with respect to the transposition of $(x,\ y)$ and $(x_0,\ y_0)$, is valid.

3. Boundary-contact conditions. The formal application of the boundary-contact operators $R_\beta{}^\alpha$, $S_\beta{}^\alpha$ to Q_α results in divergent integrals. Passing from integration to summation does not eliminate divergencies. Below we present a method for regularizing these integrals with the use of the following restrictions on L_α and $R_\beta{}^\alpha$, $S_\beta{}^\alpha$:

$$r_\beta{}^\alpha(\lambda)\, l_\alpha(\lambda) - r_\beta{}^{o\alpha}(\lambda)\, l_\alpha{}^o(\lambda) = O(\lambda^{N_\alpha}) \tag{3.1}$$

$$s_\beta{}^\alpha(\lambda)\, n_3(-\lambda) - s_\beta{}^\alpha(-\lambda)\, n_3(\lambda) = O(\lambda^{N_s}) \tag{3.2}$$

$$s_\beta{}^{o\alpha}(\lambda)\, n_3(-\lambda) - s_\beta{}^{o\alpha}(-\lambda)\, n_3(\lambda) = O(\lambda^{N_s}) \quad (\alpha = 1, 2)$$

where

$$r_\beta{}^\alpha(\lambda) = -\gamma r_{\beta 1}{}^\alpha(\lambda) + r_{\beta 2}{}^\alpha(\lambda),\quad r_\beta{}^{o\alpha}(\lambda) = \gamma r_{\beta 1}{}^\alpha(\lambda) + r_{\beta 2}{}^\alpha(\lambda)$$

$$s_\beta{}^\alpha(\lambda) = \lambda s_{\beta 1}{}^\alpha(\gamma) + s_{\beta 2}{}^\alpha(\gamma),\quad s_\beta{}^{o\alpha}(\lambda) = \lambda s_{\beta 1}{}^\alpha(-\gamma) + s_{\beta 2}{}^\alpha(-\gamma)$$

Similar formulas make possible the regularization of boundary-contact integrals for semi-infinite plates that are continuations of each other [1], and also for plates joined at a right angle [2].

We restrict our analysis to $R_\beta{}^1 Q_1$ and $S_\beta{}^1 Q_1$, since the case of $R_\beta{}^2 Q_2$ and $S_\beta{}^2 Q_2$ can be investigated similarly.

Using the identity

$$\overset{o1}{r_\beta}\, l_2 e^{\gamma h} - r_\beta{}^1 l_2{}^o e^{-\gamma h} = \frac{2\gamma r_\beta{}^1 \overset{o1}{r_\beta} D(\lambda)}{l_1 r_\beta{}^1 + l_1{}^o \overset{o1}{r_\beta}} + (r_\beta{}^1 l_2{}^o e^{-\gamma h} + \overset{o1}{r_\beta}\, l_2 e^{\gamma h})\, \frac{l_1{}^o \overset{o1}{r_\beta} - l_1 r_\beta{}^1}{l_1 r_\beta{}^1 + l_1{}^o \overset{o1}{r_\beta}} \tag{3.3}$$

we obtain

$$R_\beta{}^1 Q_1(0, 0) = \frac{1}{2\pi} \int\limits_\Lambda \frac{r_\beta{}^1 \overset{o1}{r_\beta} \Phi_1(\lambda^2)}{n_3 (l_1 r_\beta{}^1 + l_1{}^o \overset{o1}{r_\beta})}\, e^{i0\lambda} \lambda\, d\lambda + \tag{3.4}$$

$$\frac{1}{4\pi} \int\limits_\Lambda \frac{(\overset{o1}{r_\beta}\, l_2 e^{\gamma h} + r_\beta{}^1 l_2{}^o e^{-\gamma h})(\overset{o1}{r_\beta}\, l_1{}^o - l_1 r_\beta{}^1)\, \Phi_1(\lambda^2)}{\gamma n_3 D(\lambda)\, (l_1{}^o \overset{o1}{r_\beta} + l_1 r_\beta{}^1)}\, e^{i0\lambda} \lambda\, d\lambda$$

We transform contour Λ in (3.4) into the new contour Λ_1 which bypasses the roots of expression $l_1{}^o(\lambda)\, r_\beta{}^{o1}(\lambda) + l_1(\lambda)\, r_\beta{}^1(\lambda)$, from above so as not to intersect the roots of $n_3(\lambda)\, D(\lambda)$. As the result, the first integral over the contour Λ_1 in (3.4) vanishes and $R_\beta{}^1 Q_1(0, 0)$ reduces to the second integral over the contour Λ_1 which, according to estimate (3.1), is convergent, and the factor $e^{i0\lambda}$ which defines the nature of the passing to the limit can be omitted. By substituting a series in residues for the integral (3.4) we obtain the following expression:

$$R_\beta{}^1 Q_1(0, 0) = \frac{i}{2} \sum_s \frac{(\overset{o1}{r_\beta}\, l_2 e^{\gamma h} + r_\beta{}^1 l_2{}^o e^{-\gamma h})(\overset{o1}{r_\beta}\, l_1{}^o - r_\beta{}^1 l_1)\, \lambda \Phi(\lambda^2)}{\gamma n_3 D'(\lambda)\, (\overset{o1}{r_\beta}\, l_1{}^o + r_\beta{}^1 l_1)} \Bigg|_{\lambda = \lambda_s}$$

where, as in (2.13), summation is extended over the roots of $D(\lambda)$ that lie above Λ.

To regularize $S_\beta{}^1 Q_1(0, 0)$ it is sufficient to separate the even part with respect to the variable λ in the integrand obtained by the application of operator $S_\beta{}^1$. This yields

$$S_\beta{}^1 Q_1(0, 0) = \frac{1}{8\pi} \{ l_2 e^{\gamma h} [s_\beta{}^1(\lambda)\, n_3(-\lambda) - s_\beta{}^1(-\lambda)\, n_3(\lambda)] -$$

$$l_2{}^o e^{\gamma h} [\overset{o1}{s_\beta}(\lambda)\, n_3(-\lambda) - s_\beta{}^1(-\lambda)\, n_3(\lambda)] \}\, \frac{\lambda \Phi_1(\lambda^2)\, d\lambda}{\gamma n_3(\lambda)\, n_3(-\lambda)\, D(\lambda)}$$

where the integrals according to estimate (3.2) are convergent. These can also be presented in the form of series in residues with the summation extended over the roots of

$D(\lambda)$ lying above the contour Λ, and also over the roots of $n_3(-\lambda)$.

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DETERMINATION OF FREQUENCIES OF NATURAL VIBRATIONS OF CIRCULAR PLATES

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A method to construct an asymptotic process to find the axisymmetric vibration frequencies of a circular plate is proposed. Cases of symmetric vibrations relative to the middle surface (tension-compression vibrations) and of antisymmetric (bending) vibrations are considered.

The asymptotic process for a plate with free endfaces has been studied in detail under mixed boundary conditions on the side surface. This problem can be considered as a model on which the practical convergence of the method proposed is analyzed and the accuracy of finding the frequencies at each step of the process is estimated. Furthermore, problems about the natural vibrations of a circular plate under other boundary conditions on the side surface, hinged-support and rigidly fixing, are solved by the proposed method.

The purpose of this investigation is to develop a method of determining the natural vibration frequencies of a "medium" thickness plate. The question of finding the higher frequencies, even for thin plates, as well as the lowest vibration frequencies of medium thickness plates cannot be solved within the framework of existing applied theories. Hence, it is interesting to formulate a sequence of approximate theories which would permit determination of any, previously assigned, number of the first frequencies with sufficient accuracy for medium thicknesses.

1. The problem concerns the natural vibrations of a circular plate under the following boundary conditions:

$$\sigma_z = \tau_{rz} = 0, \quad z = \pm h \tag{1.1}$$

$$u_r = \tau_{rz} = 0, \quad r = a \tag{1.2}$$

Here a is the plate radius and $2h$ is its thickness. Let us construct the solution in the form

$$u_r = U(\rho, \zeta) e^{i\omega t}, \quad w = W(\rho, \zeta) e^{i\omega t}, \quad \rho = \frac{r}{a}, \quad \zeta = \frac{z}{h} \tag{1.3}$$

Satisfying the system of Lamé differential equations and the boundary conditions (1.1)